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On the Spin-Boson Model

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The existence and uniqueness of ground states of the spin-boson Hamiltonian H_{SB} are considered. The main results in the case of massive bosons include: (i)(existence) there exists a ground state *without restriction for the strength of the coupling constant*; (ii)(uniqueness) under a mild (nonperturbative) condition for the parameters contained in H_{SB}, H_{SB} has only one ground state ; (iii) (degeneracy) under a certain condition for the parameters of H_{SB} which is weaker than that of (ii), the number of the ground states is at most two. In the case of massless bosons, the existence of a ground state of H_{SB} is shown as a limit of ground states of the massive case. The methods used are *nonperturbative*. A generalization of the model is proposed.

Contents

1. Introduction and the main results

The spin-boson model, which describes a two-level quantum system coupled to a quantized Bose field, has been investigated as a simplified model for atomic systems interacting

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with a quantized radiation or phonon field ([1, 2, 5, 6, 7, 9, 14] and references therein). The ground states of the model are of particular interest. Spohn [14] discussed properties of ground states defined as zero-temperature limits of positive temperature equilibrium states. Analysis related to the work of Spohn was made by Amann [1] in terms of the notion of algebraic ground states, although it treats only a discrete version of the model. Recently attention has been paid to the ground states as the eigenvectors of the Hamiltonian H_{SB} of the model with eigenvalue equal to the infimum of its spectrum to analyze spectral properties of H_{SB} and the process of radiative decay in the model [8, 9]. In [8] Hübner and Spohn showed that, under certain conditions for the dispersion ω for bosons, the coupling function, the coupling constant α and the spectral gap μ of the unperturbed two-level system, there exists a unique ground state of H_{SB} and identify the spectrum of H_{SB}.

In this paper we focus our attention on the existence and uniqueness of ground states of the spin-boson Hamiltonian H_{SB}. We first consider the case where the bosons are massive (i.e., $m := \inf_k \omega(k) > 0$) and show that, *as far as the existence of the ground states is concerned, no restriction is needed for the coupling constant* α, which greatly improves the result on the existence of ground states in [8] (in the massive case). The basic idea to do it is as follows: we first do a unitary transformation for H_{SB} to convert it to an operator more tractable in a sense and then apply the method of constructive quantum field theory [7] to the latter operator. Moreover, by employing the min-max principle, under an additional condition for the parameters m, μ and α, which is nonperturbative, we show that H_{SB} has a unique ground state. We also suggest the possibility for H_{SB} to have degenerate ground states by showing that, under a weaker condition for m, μ and α, there exist at most two ground states of H_{SB}. In the case of massless bosons (i.e., $m = 0$), we construct a ground state as a weak limit of ground states in the massive case.

We now describe our main results. For mathematical generality, we consider the situation where bosons move in the ν-dimensional Euclidean space $\mathbb{R}^\nu$ with $\nu \geq 1$. We take the Hilbert space of bosons to be

$$\mathcal{F} = \mathcal{F}(L^2(\mathbb{R}^\nu)) = \bigoplus_{n=0}^{\infty} \left[\otimes_{\mathrm{s}}^n L^2(\mathbb{R}^\nu)\right], \tag{1.1}$$

the symmetric Fock space over $L^2(\mathbb{R}^\nu)$ ($\otimes_{\mathrm{s}}^n \mathcal{K}$ denotes the n-fold symmetric tensor product of a Hilbert space $\mathcal{K}$, $\otimes_{\mathrm{s}}^0 \mathcal{K} := \mathbb{C}$). Let ω and λ be functions on $\mathbb{R}^\nu$ satisfying the following conditions

(A.1) For all $k \in \mathbb{R}^\nu$, $\omega(k) \geq 0$ and there exist constants $\gamma > 0$ and $C > 0$ such that

$$|\omega(k) - \omega(k')| \leq C|k - k'|^\gamma, \quad k, k' \in \mathbb{R}^\nu. \tag{1.2}$$

(A.2) The function λ is real-valued and continuous with λ, $\lambda/\sqrt{\omega}$, $\lambda/\omega \in L^2(\mathbb{R}^\nu)$ and there exist constants $q > \nu/2$ and $K_0 > 0$ such that, for all $|k| \geq K_0$,

$$\left|\frac{\lambda(k)}{\omega(k)}\right| \leq \frac{D}{1 + |k|^q}$$

with D a constant (which may depend on q and K_0).

Throughout this paper, we assume (A.1) and (A.2).

A typical example of ω satisfying (A.1) is $\omega(k) = \sqrt{|k|^2 + m_0^2}$ with $m_0 \geq 0$ a constant.

We denote by $d\Gamma(\omega)$ the second quantization of the multiplication operator ω on $L^2(\mathbb{R}^\nu)$ and set

$$H_b = d\Gamma(\omega) = \int d^\nu k \omega(k) a(k)^* a(k), \tag{1.3}$$

where $a(k)$ is the operator-valued distribution kernel of the smeared annihilation operator $a(f) = \int a(k) f(k)^* d^\nu k$ ($f \in L^2(\mathbb{R}^\nu)$) on $\mathcal{F}$ (f^* denotes the complex conjugate of f). The Hamiltonian of the spin-boson model is defined by

$$H_{\mathrm{SB}} = \frac{1}{2}\mu\sigma_z \otimes I + I \otimes H_b + \alpha\sigma_x \otimes (a(\lambda)^* + a(\lambda)) \tag{1.4}$$

acting in the Hilbert space

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathcal{F} = \mathcal{F} \oplus \mathcal{F}, \tag{1.5}$$

where σ_x, σ_z are the standard Pauli matrices, $\mu > 0$ and $\alpha \in \mathbb{R}$ are constants denoting the spectral gap of the unpertubed two-level system and the coupling constant, respectively, and I denotes identity.

For a linear operator T on a Hibert space, we denote its domain by $D(T)$. It is well known that H_{SB} is self-adjoint with $D(H_{\mathrm{SB}}) = D(I \otimes H_b)$ and

$$H_{\mathrm{SB}} \geq -\frac{\mu}{2} - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_{L^2}^2, \tag{1.6}$$

where $\|\cdot\|_{L^2}$ denotes the norm of $L^2(\mathbb{R}^\nu)$.

For a self-adjoint operator T bounded from below, we denote by $E(T)$ the infimum of the spectrum $\sigma(T)$ of T:

$$E(T) = \inf \sigma(T). \tag{1.7}$$

In this paper, an eigenvector of T with eigenvalue $E(T)$ is called a *ground state of* T (if it exists). We say that T has a (resp. uniuqe) ground state if $\dim\ker(T - E(T)) \geq 1$ (resp. $\dim\ker(T - E(T)) = 1$).

The following estimate for $E(H_{\mathrm{SB}})$ is well known (see (2.10) below) :

$$-\frac{\mu}{2} - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_{L^2}^2 \leq E(H_{\mathrm{SB}}) \leq -\frac{\mu}{2} e^{-2\alpha^2 \|\lambda/\omega\|_{L^2}^2} - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_{L^2}^2. \tag{1.8}$$

Let

$$m := \inf_{k \in \mathbb{R}^\nu} \omega(k) \tag{1.9}$$

We have the following result on the existence of ground states of H_{SB} :

THEOREM 1.1. *Assume (A.1), (A.2) and $m > 0$. Then $H_{\rm SB}$ has purely discrete spectrum in the interval $[E(H_{\rm SB}), E(H_{\rm SB}) + m)$. In particular, $H_{\rm SB}$ has a ground state.*

Remark: Theorem 1.1 implies that, under the same assumption, $\inf \sigma_{\rm ess}(H_{\rm SB}) \geq E(H_{\rm SB}) + m$, where $\sigma_{\rm ess}(\cdot)$ denotes essential spectrum, i.e., $H_{\rm SB}$ has a spectral gap. In a forthcoming paper, we shall show that, in fact, $\sigma_{\rm ess}(H_{\rm SB}) = [E(H_{\rm SB}) + m, \infty)$.

To state our result on the uniqueness of ground states, we introduce

$$K_\varepsilon(\alpha,\mu) = \min\left\{m(1-\varepsilon), \frac{\mu}{2}\right\} - \frac{4\alpha^2\mu^2}{\varepsilon}\left\|\frac{\lambda}{\omega\sqrt{\omega}}\right\|_{L^2}^2 - 2|\alpha|\mu\left\|\frac{\lambda}{\omega}\right\|_{L^2}, \tag{1.10}$$

with λ such that $\lambda/\omega\sqrt{\omega} \in L^2(\mathbb{R}^\nu)$.

Remark: If $m > 0$, then $\lambda \in L^2(\mathbb{R}^\nu)$ implies that, for all $s > 0$, $\lambda/\omega^s \in L^2(\mathbb{R}^\nu)$.

THEOREM 1.2. *Assume (A.1), (A.2) and $m > 0$. Suppose that*

$$\sup_{0<\varepsilon<1} K_\varepsilon(\alpha,\mu) > \frac{\mu}{2}\left(1 - e^{-2\alpha^2\|\lambda/\omega\|_{L^2}^2}\right) \tag{1.11}$$

Then $H_{\rm SB}$ has a unique ground state.

Remark: By applying regular perturbation theory (e.g., [12, Chapt.XII]), one can easily show that there exists a constant $\alpha_0 > 0$ such that, for all $\alpha \in (-\alpha_0, \alpha_0)$, $H_{\rm SB}$ has a unique ground state. For arbitrarily fixed $m > 0$ and $\mu > 0$, (1.11) is satisfied if $|\alpha|$ is sufficiently small. Thus Theorem 1.2 may be regarded as a result which improves the one obtained by regular perturbation theory. Note that (1.11) is a nonperturbative estimate in α, since the right hand side (RHS) of (1.11) is non-polynomial in α. We believe that (1.11) is a relatively good estimate to ensure the uniqueness of ground states of $H_{\rm SB}$ (see th proof of Theorem 1.2 in §5.2).

As is easily seen, in the case $\mu = 0$, $H_{\rm SB}$ has two-fold degenerate ground states. This fact suggests that $H_{\rm SB}$ with $\mu > 0$ also may have denenerate ground states. In this respect, we have the following result:

THEOREM 1.3. *Assume (A.1), (A.2) and $m > 0$. Suppose that*

$$m > \frac{\mu}{2}\left(1 - e^{-2\alpha^2\|\lambda/\omega\|_{L^2}^2}\right). \tag{1.12}$$

Then the following (a) and (b) hold:

(a) *There are at most two eigenvalues (counting multiplicity) of $H_{\rm SB}$ in the interval $[E(H_{\rm SB}), -\frac{\mu}{2}e^{-2\alpha^2\|\lambda/\omega\|_{L^2}^2} - \alpha^2\|\lambda/\sqrt{\omega}\|_{L^2}^2]$.*

(b) *The Hamiltonian $H_{\rm SB}$ has at most two ground states, i.e., $\dim\ker(H_{\rm SB} - E(H_{\rm SB})) \leq 2$.*

Remark: Condition (1.11) implies (1.12), i.e., the latter condition is weaker than the former.

In the case of *massless bosons*, we have the following result on the existence of ground states of H_{SB}:

THEOREM 1.4. *Assume (A.1), (A.2) and $m = 0$. Suppose, in addition, that $\omega\lambda \in L^2(\mathbb{R}^\nu)$ and*

$$|\alpha| < \frac{1}{\|\lambda/\omega\|_{L^2}}. \tag{1.13}$$

Then H_{SB} has a ground state.

Remark: To our best knowledge, Theorem 1.4 is the first which establishes the existence of ground states of the spin-boson Hamiltonian H_{SB} in the case of *massless bosons*.

The present paper is organized as follows. In Section 2 we review some basic facts on the spin-boson Hamiltonian H_{SB}. We recall a well known unitary transformation which converts H_{SB} to an operaotr H simpler in a sense. We analyze the operator H. To prove the exsitence of ground states of H, we introduce in Section 3 a finite volume approximation H_V $(V > 0)$ for H. In Section 4 we prove that H_V converges to H in the norm resolvent sense as $V \to \infty$. In Section 5 we prove Theorems 1.1 – 1.4. In the last section we propose a generalization of the model.

2. Some basic facts

It is well known that, for all $f \in L^2(\mathbb{R}^\nu)$, the operator

$$P(f) := i\{a(f)^* - a(f)\} \tag{2.1}$$

is essentially self-adjoint on the finite particle subspace

$$\mathcal{F}_0 = \{\Psi = \{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{F} |\ \text{only finitely many } \Psi_n\text{'s are not zero}\}. \tag{2.2}$$

We denote the closure of $P(f)$ by the same symbol. Let

$$U_\pm = e^{\pm i\alpha P(\lambda/\omega)}. \tag{2.3}$$

Then

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} U_+ & U_- \\ U_+ & -U_- \end{pmatrix} \tag{2.4}$$

is unitary on $\mathcal{H}$. Moreover, we have

$$U^{-1} H_{\mathrm{SB}} U = H - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_{L^2}^2 \tag{2.5}$$

with

$$H = I \otimes H_b + \frac{\mu}{2}(A \otimes U_+^2 + A^* \otimes U_-^2), \tag{2.6}$$

where

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.7}$$

Based on (2.5), we shall consider, instead of H_{SB}, the operator H defined by (2.6). An advantage of this approach is in that the perturbation term

$$H_I := \frac{\mu}{2}(A \otimes U_+^2 + A^* \otimes U_-^2) \tag{2.8}$$

of H is a bounded self-adjoint operator. The operator norm $||H_I||$ of H_I can be exactly computed:

LEMMA 2.1. *We have*

$$||H_I|| = \frac{\mu}{2}. \tag{2.9}$$

PROOF: We need only to use the relation $H_I = \frac{\mu}{2}U^{-1}(\sigma_z \otimes I)U$ and the fact $||\sigma_z \otimes I|| = 1$. ∎

It follows from (2.9) and the variational principle (cf. [2, 4]) that

$$-\frac{\mu}{2} \le E(H) \le -\frac{\mu}{2}e^{-2\alpha^2||\lambda/\omega||^2_{L^2}} < 0. \tag{2.10}$$

LEMMA 2.2. *Assume, in addition to (A.1) and (A.2), that $\omega\lambda \in L^2(\mathbb{R}^\nu)$. Let Ψ be any eigenvector of H_{SB}. Then $\Psi \in D((I \otimes H_b)^{3/2})$.*

PROOF: By the assumption, we have $H_{\mathrm{SB}}\Psi = E\Psi, \Psi \in D(H_{\mathrm{SB}}) = D(I \otimes H_b)$ with E an eigenvalue of H_{SB}. Hence

$$(I \otimes H_b)\Psi = E\Psi - \frac{\mu}{2}(\sigma_z \otimes I)\Psi - \alpha\sigma_x \otimes [a(\lambda)^* + a(\lambda)]\Psi.$$

The vectors on the RHS except for the last one is in $D(I \otimes H_b)$. We denote by $a(\cdot)^\#$ either $a(\cdot)^*$ or $a(\cdot)$. It is known that, if $\omega f, f/\sqrt{\omega} \in L^2(\mathbb{R}^\nu)$, then $a^\#(f)$ maps $D(H_b)$ into $D(H_b^{1/2})$[3, Lemma 2.4]. Hence $\sigma_x \otimes [a(\lambda)^* + a(\lambda)]\Psi \in D((I \otimes H_b)^{1/2})$. Thus we conclude that $(I \otimes H_b)\Psi \in D((I \otimes H_b)^{1/2})$, which implies the desired result. ∎

Let

$$N = d\Gamma(I) = \int d^\nu k a(k)^* a(k), \tag{2.11}$$

the number operator on $\mathcal{F}$.

In general we denote by $(\ \cdot\ ,\ \cdot\ )_{\mathcal{K}}$ and $||\cdot||_{\mathcal{K}}$ the inner product (complex linear in the second variable) and the norm of a Hilbert space $\mathcal{K}$, respectively, but, we sometimes omit the subscript $\mathcal{K}$ if there is no danger of confusion.

LEMMA 2.3. *Assume, in addition to (A.1) and (A.2), that $\omega\lambda \in L^2(\mathbb{R}^\nu)$. Then, for every normalized ground state Ω of H_{SB},*

$$(\Omega, I \otimes N\Omega)_{\mathcal{H}} \leq \alpha^2 \left\| \frac{\lambda}{\omega} \right\|_{L^2}^2 . \tag{2.12}$$

PROOF: Let f be a function such that $\omega f, f/\sqrt{\omega} \in L^2(\mathbb{R}^\nu)$ (then $f \in L^2(\mathbb{R}^\nu)$). It follows from Lemma 2.2 and a mapping property of $a(f)^\#$ [3, Lemma 2.3] that $a(f)\Omega \in D(I \otimes H_b) = D(H_{\mathrm{SB}})$. Since $H_{\mathrm{SB}} - E(H_{\mathrm{SB}}) \geq 0$, we have

$$\begin{aligned} 0 &\leq (I \otimes a(f)\Omega, [H_{\mathrm{SB}} - E(H_{\mathrm{SB}})]\, I \otimes a(f)\Omega) \\ &= (I \otimes a(f)\Omega, [H_{\mathrm{SB}}, I \otimes a(f)]\Omega) \\ &= (I \otimes a(f)\Omega, (-I \otimes a(\omega f) - \alpha(\sigma_x \otimes I)(f,\lambda)_{L^2})\Omega). \end{aligned}$$

Hence

$$(\Omega, I \otimes a(f)^* a(\omega f)\Omega) + \alpha (f,\lambda)_{L^2} (\sigma_x \otimes a(f)\Omega, \Omega) \leq 0. \tag{2.13}$$

There exists a sequence $\{f_n\}_{n=1}^\infty$ of functions such that $\omega f_n, f_n/\sqrt{\omega} \in L^2(\mathbb{R}^\nu)$ for all $n \geq 1$ and $\{\sqrt{\omega} f_n\}_{n=1}^\infty$ is a complete orthonormal system of $L^2(\mathbb{R}^\nu)$. By (2.13), we have for all $N = 1, 2, 3, \cdots$

$$\sum_{n=1}^N (\Omega, I \otimes a(f_n)^* a(\omega f_n)\Omega) + \alpha(\sigma_x \otimes a(F_N)\Omega, \Omega) \leq 0,$$

where $F_N = \sum_{n=1}^N (f_n, \lambda)_{L^2} f_n$. It is not so difficult to show that

$$\lim_{N\to\infty} \sum_{n=1}^N (\Omega, I \otimes a(f_n)^* a(\omega f_n)\Omega) = (\Omega, I \otimes N\Omega),$$

$$\lim_{N\to\infty} (\sigma_x \otimes a(F_N)\Omega, \Omega) = \left(\sigma_x \otimes a\left(\frac{\lambda}{\omega}\right)\Omega, \Omega\right).$$

Hence $(\Omega, I \otimes N\Omega) + \alpha\left(\sigma_x \otimes a\left(\frac{\lambda}{\omega}\right)\Omega, \Omega\right) \leq 0$. Since $(\Omega, I \otimes N\Omega) \geq 0$, it follows that $\left(\sigma_x \otimes a\left(\frac{\lambda}{\omega}\right)\Omega, \Omega\right)$ is real and

$$(\Omega, I \otimes N\Omega) \leq -\alpha \left(\sigma_x \otimes a\left(\frac{\lambda}{\omega}\right)\Omega, \Omega\right). \tag{2.14}$$

Applying the well known estimate

$$\|a(f)\Psi\|_{\mathcal{F}} \leq \|f\|_{L^2} \|N^{1/2}\Psi\|_{\mathcal{F}}, \quad f \in L^2(\mathbb{R}^\nu), \Psi \in D(N^{1/2}), \tag{2.15}$$

to the RHS of (2.14), we obtain

$$(\Omega, I \otimes N\Omega) \leq |\alpha| \left\| \frac{\lambda}{\omega} \right\|_{L^2} \|(I \otimes N)^{1/2}\Omega\|,$$

which implies (2.12). ∎

Inequality (2.12) gives an upper bound for the mean of boson numbers in any normalized ground state of H_{SB}. Note that inequality (2.12) is independent of whether bosons are massive or massless.

3. A finite volume approximation

Let $V > 0$ be a parameter and

$$\Gamma_V = \frac{2\pi\mathbb{Z}^\nu}{V} = \left\{ k = (k_1, \cdots, k_\nu) \middle| k_j = \frac{2\pi n_j}{V}, n_j \in \mathbb{Z}, j = 1, \cdots, \nu \right\}. \tag{3.1}$$

Let

$$\mathcal{F}_V = \mathcal{F}(\ell^2(\Gamma_V)) = \bigoplus_{n=0}^{\infty} [\otimes_{\mathrm{s}}^n \ell^2(\Gamma_V)] \tag{3.2}$$

the symmetric Fock space over $\ell^2(\Gamma_V)$, which describes state vectors of bosons in the finite box $[-V/2, V/2]^\nu$. Each element Ψ in $\otimes_{\mathrm{s}}^n \ell^2(\Gamma_V)$ can be identified with a piecewise constant function in $\otimes_{\mathrm{s}}^n L^2(\mathbb{R}^\nu)$ which is a constant on each cube of volume $(2\pi/V)^{n\nu}$ centered about a lattice point

$$(k_1, \cdots, k_n) \in \Gamma_V \times \cdots \times \Gamma_V = \Gamma_V^n.$$

With this identification, $\mathcal{F}_V$ is regarded as a closed subspace of $\mathcal{F}$.

For each $k = (k_1, \cdots, k_\nu) \in \Gamma_V$, we define a function $\chi_{k,V}$ on $\mathbb{R}^\nu$ by

$$\chi_{k,V}(\ell) = \chi_{[k_1 - \frac{\pi}{V}, k_1 + \frac{\pi}{V}]}(\ell_1) \cdots \chi_{[k_\nu - \frac{\pi}{V}, k_\nu + \frac{\pi}{V}]}(\ell_\nu), \quad \ell = (\ell_1, \cdots, \ell_\nu) \in \mathbb{R}^\nu, \tag{3.3}$$

where $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a, b]$. We introduce

$$a_V(k) := \left(\frac{V}{2\pi}\right)^{\nu/2} a(\chi_{k,V}) = \left(\frac{V}{2\pi}\right)^{\nu/2} \int_{[-\pi/V, \pi/V]^\nu} a(k + \ell) d\ell. \tag{3.4}$$

It is easy to see that, for all $k, \ell \in \Gamma_V$,

$$[a_V(k), a_V(\ell)^*] = \delta_{k\ell}, \quad [a_V(k), a_V(\ell)] = 0, \tag{3.5}$$

on $\mathcal{F}_0$.

We define

$$\omega_V(k) = \omega(k_V), \quad k \in \mathbb{R}^\nu, \tag{3.6}$$

with k_V a lattice point closed to k:

$$k_V \in \Gamma_V, \quad |k_j - (k_V)_j| \le \frac{\pi}{V}, \quad j = 1, \cdots, \nu. \tag{3.7}$$

Let

$$H_{b,V} := d\Gamma(\omega_V) = \int d^\nu k \omega_V(k) a(k)^* a(k). \tag{3.8}$$

LEMMA 3.1. *We have*

$$D(H_{b,V}) = D(H_b) \tag{3.9}$$

and there exists a constant $c > 0$ independent of V such that, for all $\Psi \in D(N)$,

$$||(H_b - H_{b,V})\Psi|| \leq \frac{c}{V^\gamma}||N\Psi||. \tag{3.10}$$

PROOF: By (1.2) and (3.7), we have for all $k \in \mathbb{R}^\nu$, $|\omega(k) - \omega(k_V)| \leq c/V^\gamma$ with $c = C\pi^\gamma \nu^{\gamma/2}$, from which (3.9) and (3.10) follow. ∎

The following fact is well known:

LEMMA 3.2. *The operator $H_{b,V}$ is reduced by $\mathcal{F}_V$ and*

$$H_{b,V} \restriction \mathcal{F}_V = \sum_{k\in\Gamma_V} \omega(k) a_V(k)^* a_V(k).$$

For notational simplicity, we set

$$g(k) = \frac{\alpha\lambda(k)}{\omega(k)}. \tag{3.11}$$

For $K > 0$, we define a function $g_{K,V}$ on $\mathbb{R}^\nu$ by

$$g_{K,V} = \sum_{k\in\Gamma_V, |k_j|\leq K, j=1,\cdots,\nu} g(k)\chi_{k,V}.$$

LEMMA 3.3. *The function $g_{K,V}$ converges in $L^2(\mathbb{R}^\nu)$ as $K \to \infty$.*

PROOF: For a constant $K > 0$, we put

$$S_{K,V} = \sum_{k\in\Gamma_V, |k_j|\leq K, j=1,\cdots,\nu} \left(\frac{2\pi}{V}\right)^\nu |g(k)|^2$$

Then, by the growth condition for λ/ω in (A.2), we have

$$\begin{aligned} S_{K,V} &\leq \sum_{k\in\Gamma_V, |k|\leq K_0} \left(\frac{2\pi}{V}\right)^\nu |g(k)|^2 + \alpha^2 D^2 \sum_{k\in\Gamma_V, |k|\geq K_0} \left(\frac{2\pi}{V}\right)^\nu \frac{1}{(1+|k|^q)^2} \\ &\leq \sum_{k\in\Gamma_V, |k|\leq K_0} \left(\frac{2\pi}{V}\right)^\nu |g(k)|^2 + \alpha^2 D^2 \int_{\mathbb{R}^\nu} \frac{1}{(1+|k|^q)^2} dk < \infty. \end{aligned}$$

Hence $S_{K,V}$ is uniformly bounded in K. Since $S_{K,V}$ is monotone non-decreasing in K, it follows that the infinite series $S_V := \sum_{k\in\Gamma_V} \left(\frac{2\pi}{V}\right)^\nu |g(k)|^2$ converges. Let $K' \geq K$. Then we have $(g_{K,V}, g_{K',V})_{L^2} = S_{K,V} \to S_V$ $(K \to \infty)$, which implies that $\{g_{K,V}\}_K$ is a Cauchy net. ∎

We write

$$g_V = L^2 - \lim_{K\to\infty} g_{K,V} = \sum_{k\in\Gamma_V} g(k)\chi_{k,V}. \tag{3.12}$$

Then we have

$$P(g_V) = i\left(\frac{2\pi}{V}\right)^{\nu/2} \sum_{k\in\Gamma_V} g(k)(a_V(k)^* - a_V(k)) \tag{3.13}$$

on $\mathcal{F}_0$.

Let

$$U_\pm(V) = e^{\pm iP(g_V)}. \tag{3.14}$$

and

$$H_V = I \otimes H_{b,V} + \frac{\mu}{2}\{A \otimes U_+(V)^2 + A^* \otimes U_-(V)^2). \tag{3.15}$$

LEMMA 3.4. *The operator H_V is self-adjoint with $D(H_V) = D(I \otimes H_b)$ and bounded from below with*

$$H_V \geq -\frac{\mu}{2}. \tag{3.16}$$

PROOF: Since the operator

$$H_I(V) := \frac{\mu}{2}\{A \otimes U_+(V)^2 + A^* \otimes U_-(V)^2) \tag{3.17}$$

is bounded, the Kato-Rellich theorem gives the self-adjointness of H_V with $D(H_V) = D(I \otimes H_{b,V}) = D(I \otimes H_b)$ (Lemma 3.1). Inequality (3.16) follows from the fact $||H_I(V)|| = \frac{\mu}{2}$, which can be proven in the same way as in Lemma 2.1. ∎

In the next section, we show that H_V is a finite volume approximation for H in a suitable sense.

4. Convergence of the finite volume approximation

In this section we prove the following theorem:

THEOREM 4.1. *For all $z \in \mathbb{C}$ with $Im\, z \neq 0$ or $z < -\mu/2$,*

$$\lim_{V\to\infty} ||(H_V - z)^{-1} - (H - z)^{-1}|| = 0. \tag{4.1}$$

To prove this theorem, we prepare some lemmas.

LEMMA 4.2.

$$\lim_{V\to\infty} ||g_V - g||_{L^2} = 0. \tag{4.2}$$

PROOF: By the growth condition for λ/ω in (A.2), one can easily show that

$$||g_V||_{L^2}^2 = \sum_{k\in\Gamma_V} \left(\frac{2\pi}{V}\right)^{\nu} |g(k)|^2 \to \int_{\mathbb{R}^\nu} d^\nu k |g(k)|^2 = ||g||_{L^2}^2 \quad (V\to\infty). \tag{4.3}$$

Let $f \in C_0^\infty(\mathbb{R}^\nu)$ and $\operatorname{supp} f \subset \{k \in \mathbb{R}^\nu |\ |k_j| \le K_f, j = 1,\cdots,\nu\}$ with a constant K_f. Then we have

$$(f, g_V)_{L^2} = \sum_{\ell\in\Gamma_V} \left(\frac{2\pi}{V}\right)^{\nu} f(\ell)^* g(\ell) + I_V,$$

where

$$I_V = \sum_{\ell\in\Gamma_V, |\ell_j|\le K_f, j=1,\cdots,\nu} g(\ell) \int_{[\ell_1-\frac{\pi}{V},\ell_1+\frac{\pi}{V}]\times\cdots\times[\ell_\nu-\frac{\pi}{V},\ell_\nu+\frac{\pi}{V}]} [f(k)^* - f(\ell)^*] d^\nu k.$$

Since f is uniformly continuous, for any $\varepsilon > 0$, there exists a constant $V_0 > 0$ such that, if $|k_j - \ell_j| \le \pi/V_0$, then $|f(k) - f(\ell)| \le \varepsilon$. Hence, for all $V \ge V_0$, we have $|I_V| \le D_V\varepsilon$, where $D_V = \sum_{\ell\in\Gamma_V, |\ell_j|\le K_f, j=1,\cdots,\nu} \left(\frac{2\pi}{V}\right)^\nu g(\ell)$. Note that

$$\lim_{V\to\infty} D_V = D := \int_{[-K_f,K_f]^\nu} |g(k)| d^\nu k \le \left(\int_{[-K_f,K_f]^\nu} |g(k)|^2 d^\nu k\right)^{1/2} (2K_f)^{\nu/2} < \infty.$$

Hence $\overline{\lim}_{V\to\infty} |I_V| \le D\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $\lim_{V\to\infty} I_V = 0$. Thus we obtain

$$(f, g_V)_{L^2} \to (f, g)_{L^2} \quad (V\to\infty). \tag{4.4}$$

By (4.3), (4.4) and a limiting argument using the denseness of $C_0^\infty(\mathbb{R}^\nu)$ in $L^2(\mathbb{R}^\nu)$, we obtain (4.2). ∎

We say that two self-adjoint operators T_1 and T_2 on a Hilbert space *strongly commute* if their spectral measures commute.

LEMMA 4.3. *Let T_1 and T_2 be strongly commuting self-adjoint operators on a Hilbert space. Then, for all $\psi \in D(T_1) \cap D(T_2)$,*

$$||(e^{iT_1} - e^{iT_2})\psi|| \le ||(T_1 - T_2)\psi||.$$

PROOF: Let E_j be the spectral measure of T_j. Then there exists a unique two-dimensional spectral measure E such that, for all Borel sets B_1, B_2 in $\mathbb{R}$, $E(B_1 \times B_2) = E_1(B_1)E_2(B_2)$. In terms of E, we have

$$T_j = \int \lambda_j dE(\lambda_1, \lambda_2), \quad e^{iT_j} = \int e^{i\lambda_j} dE(\lambda_1, \lambda_2), \quad j = 1, 2.$$

By the functional calculus and the inequality $|e^{ix} - e^{iy}| \le |x - y|, x, y \in \mathbb{R}$, we have for all $\psi \in D(T_1) \cap D(T_2)$

$$\begin{aligned}||(e^{iT_1} - e^{iT_2})\psi||^2 &= \int_{\mathbb{R}^2} |e^{i\lambda_1} - e^{i\lambda_2}|^2 d||E(\lambda_1, \lambda_2)\psi||^2 \\ &\le \int_{\mathbb{R}^2} |\lambda_1 - \lambda_2|^2 d||E(\lambda_1, \lambda_2)\psi||^2 \\ &= ||(T_1 - T_2)\psi||^2.\end{aligned}$$

Thus the desired result follows. ∎

LEMMA 4.4.
$$||(U_\pm(V)^2 - U_\pm^2)(N + I)^{-1/2}|| \le 4||g_V - g||. \tag{4.5}$$

PROOF: For all real-valued functions $f_1, f_2 \in L^2(\mathbb{R}^\nu)$ and all $s, t \in \mathbb{R}$, $e^{itP(f_1)}$ commutes with $e^{isP(f_2)}$ (e.g., [11, Theorem X.43]). Hence, by a general theorem (e.g., [10, Theorem VIII.13], $P(f_1)$ and $P(f_2)$ strongly commute. Applying this fact, we conclude that $P(g)$ and $P(g_V)$ strongly commute. Hence, by Lemma 4.3, we have for all $\Psi \in \mathcal{F}_0$,

$$\begin{aligned}||(U_\pm(V)^2 - U_\pm^2)\Psi|| &\le 2||(P(g_V) - P(g))\Psi|| \\ &\le 2(||a(g_V - g)\Psi|| + ||a(g_V - g)^*\Psi||).\end{aligned}$$

By (2.15) and the complementary estimate to it

$$||a(f)^*\Phi|| \le ||f||_{L^2}||(N + I)^{1/2}\Phi||, \quad \Phi \in D(N^{1/2}), f \in L^2(\mathbb{R}^\nu),$$

we obtain
$$||(U_\pm(V)^2 - U_\pm^2)\Psi|| \le 4||g_V - g|| \cdot ||(N + I)^{1/2}\Psi||.$$

Since $\mathcal{F}_0$ is a core of $N^{1/2}$, we can extend this inequality, via a simple limiting argument, to all $\Psi \in D(N^{1/2})$. Thus (4.5) follows. ∎

Proof of Theorem 4.1
We prove (4.1) in the case $\operatorname{Im} z \neq 0$ (the other case can be similarly treated). Writing

$$I \otimes H_b = H - H_I$$

and using Lemma 2.1, we have

$$||I \otimes H_b \Psi|| \leq ||H\Psi|| + \frac{\mu}{2}||\Psi||, \quad \Psi \in D(I \otimes H_b).$$

Let $L = I \otimes N + I$. By the fact that $||N\Phi|| \leq ||H_b \Phi||/m, \Phi \in D(H_b)$, we obtain

$$||(L - I)\Psi|| \leq \frac{1}{m}\left(||H\Psi|| + \frac{\mu}{2}||\Psi||\right), \quad \Psi \in D(I \otimes H_b),$$

which implies that, for all $z \in \mathbb{C} \setminus \mathbb{R}$, $L(H - z)^{-1}$ is bounded. By Lemma 3.1, $(I \otimes H_b - I \otimes H_{b,V})L^{-1}$ is bounded with

$$||(I \otimes H_b - I \otimes H_{b,V})L^{-1}|| \leq \frac{c}{V^\gamma}. \tag{4.6}$$

We write

$$\begin{aligned}(H_V - z)^{-1} - (H - z)^{-1} =& (H_V - z)^{-1}(I \otimes H_b - I \otimes H_{b,V})L^{-1}L(H - z)^{-1} \\ &+ (H_V - z)^{-1}(H_I - H_I(V))L^{-1/2}L^{1/2}(H - z)^{-1}.\end{aligned}$$

Hence

$$\begin{aligned}||(H_V - z)^{-1} - (H - z)^{-1}|| \leq \frac{1}{|\operatorname{Im} z|}\Big(&||(H_b - H_{b,V})L^{-1}|| \cdot ||L(H - z)^{-1}|| \\ &+ ||(H_I - H_I(V))L^{-1/2}|| \cdot ||L^{1/2}(H - z)^{-1}||\Big).\end{aligned}$$

We have

$$H_I - H_I(V) = \frac{\mu}{2}\{A \otimes (U_+^2 - U_+(V)^2) + A^* \otimes (U_-^2 - U_-(V)^2)\}.$$

Hence, by Lemma 4.4, $||(H_I - H_I(V))L^{-1/2}|| \leq 4\mu \cdot ||g_V - g||$, which, combined with Lemma 4.2, implies that $\lim_{V\to\infty} ||(H_I - H_I(V))L^{-1/2}|| = 0$. By (4.6), we have $\lim_{V\to\infty} ||(H_b - H_{b,V})L^{-1}|| = 0$. Thus we obtain (4.1). ∎

5. Proof of the main results

5.1. *Proof of Theorem 1.1*

Let

$$\mathcal{H}_V = \mathbb{C}^2 \otimes \mathcal{F}_V.$$

LEMMA 5.1. *The operator $H_V \restriction \mathcal{H}_V$ has purely discrete spectrum.*

PROOF: It is well known or easy to see that $I \otimes H_{b,V} \restriction \mathcal{H}_V$ has compact resolvent. Since $H_I(V)$ is bounded, it follows that $H_I(V)(I \otimes H_{b,V} + i)^{-1} \restriction \mathcal{H}_V$ is compact. Hence, by a general theorem [12, §XIII.4, Corollary 2], $\sigma_{\text{ess}}(H_V \restriction \mathcal{H}_V) = \sigma_{\text{ess}}(I \otimes H_{b,V} \restriction \mathcal{H}_V) = \emptyset$. Thus the desired result follows. ∎

LEMMA 5.2.

$$H_V \restriction \mathcal{H}_V^\perp \geq E(H_V) + m.$$

PROOF: We decompose $L^2(\mathbb{R}^\nu)$ as $L^2(\mathbb{R}^\nu) = F_{1V} \oplus F_{1V}^\perp$ with $F_{1V} = L^2(\mathbb{R}^\nu) \cap \mathcal{F}_V$. Then

$$\mathcal{F} = \mathcal{F}_V \otimes \mathcal{F}(F_{1V}^\perp) = \bigoplus_{j=0}^{\infty} \mathcal{F}^{(j)},$$

where $\mathcal{F}^{(j)} = \mathcal{F}_V \otimes [\otimes_s^j F_{1V}]$. Hence $\mathcal{F}_V^\perp = \bigoplus_{j=1}^\infty \mathcal{F}^{(j)}$ and $\mathcal{H}_V^\perp = \mathbb{C}^2 \otimes \mathcal{F}_V^\perp = \bigoplus_{j=1}^\infty \mathbb{C}^2 \otimes \mathcal{F}^{(j)}$. On each $\mathbb{C}^2 \otimes \mathcal{F}^{(j)}$, H_V has the form $S \otimes I + I \otimes T$ with $S = H_V \restriction \mathcal{H}_V$ and T is a sum of j copies of $H_{b,V}$, each acting on a single factor $F_{1V}^\perp$. Since $T \geq jm$ on $\otimes_s^j F_{1V}$, the assertion of the lemma follows. ∎

LEMMA 5.3 [13, LEMMA 4.6]. *Let T_n and T be self-adjoint operators on a Hilbert space, which are bounded from below. Suppose that $T_n \to T$ in norm resolvent sense as $n \to \infty$ and T_n has purely discrete spectrum in $[E(T_n), E(T_n) + c)$ with some constnat $c > 0$. Then, $\lim_{n\to\infty} E(T_n) = E(T)$ and T has purely discrete spectrum in $[E(T), E(T) + c)$.*

We are now ready to prove Theorem 1.1 : By Lemmas 5.1 and 5.2, H_V has purely discrete spectrum in $[E(H_V), E(H_V) + m)$. By this fact and Theorem 4.1, we can apply Lemma 5.3 to conclude that H has purely discrete spectrum in $[E(H), E(H) + m)$, which, combined with (2.5), implies Theorem 1.1.

5.2. *Proof of Theorem 1.2*

The basic idea of proof is to use the min-max principle for H [12, Theorem XIII.1]. Let

$$\mu_2(H) = \sup_{\Phi \in \mathcal{H}} U_H(\Phi)$$

with $U_H(\Phi) = \inf_{\Psi \in D(H), ||\Psi||=1, \Psi \in [\Phi]^\perp} (\Psi, H\Psi)$, where $[\Phi]^\perp = \{\Psi \in \mathcal{H} | (\Psi, \Phi) = 0\}$. We estimate $\mu_2(H)$ from below. For this purpose, we write

$$H = I \otimes H_b + \frac{\mu}{2} \sigma_x \otimes I + W,$$

where

$$W = \frac{\mu}{2}\left\{A \otimes (U_+^2 - I) + A^* \otimes (U_-^2 - I)\right\}.$$

For $\varepsilon > 0$, we set

$$D_\varepsilon(\alpha,\mu) = \frac{4\alpha^2\mu^2}{\varepsilon}\left\|\frac{\lambda}{\omega\sqrt{\omega}}\right\|_{L^2}^2 + 2|\alpha|\mu\left\|\frac{\lambda}{\omega}\right\|_{L^2}.$$

LEMMA 5.4. *For all $\varepsilon > 0$ and $\Psi \in D(I \otimes H_b)$,*

$$|(\Psi, W\Psi)| \le \varepsilon(\Psi, I \otimes H_b\Psi) + D_\varepsilon(\alpha,\mu)||\Psi||^2. \tag{5.1}$$

PROOF: By the fact $||A|| = ||A^*|| = 1$ and Lemma 4.3, we have for all $\Psi \in D(I \otimes H_b)$

$$\begin{aligned}||W\Psi|| &\le \frac{\mu}{2}\left(||I \otimes (U_+^2 - I)\Psi|| + ||I \otimes (U_-^2 - I)\Psi||\right)\\ &\le 2|\alpha|\mu||I \otimes P(\lambda/\omega)\Psi||\\ &\le 2|\alpha|\mu(||I \otimes a(\lambda/\omega)\Psi|| + ||I \otimes a(\lambda/\omega)^*\Psi||).\end{aligned}$$

On the other hand, the following estimates are well known:

$$\begin{aligned}&||a(f)\psi|| \le ||f/\sqrt{\omega}||_{L^2}||H_b^{1/2}\psi||,\\ &||a(f)^*\psi|| \le ||f/\sqrt{\omega}||_{L^2}||H_b^{1/2}\psi|| + ||f||_{L^2}||\psi||, \quad f, f/\sqrt{\omega} \in L^2(\mathbb{R}^\nu), \psi \in D(H_b^{1/2}).\end{aligned}$$

Hence

$$||W\Psi|| \le 4|\alpha|\mu\left\|\frac{\lambda}{\omega\sqrt{\omega}}\right\|_{L^2}||(I \otimes H_b)^{1/2}\Psi|| + 2|\alpha|\mu||\Psi||\left\|\frac{\lambda}{\omega}\right\|_{L^2}.$$

Using this estimate and the elementary inequality $xy \le \varepsilon x^2 + \frac{y^2}{4\varepsilon}$ holding for all $x, y, \varepsilon > 0$, we obtain (5.1). ∎

We now proceed to proof of Theorem 1.2. Let Ω_0 be the Fock vacuum in $\mathcal{F}$: $\Omega_0 = \{1, 0, 0, \cdots\}$ and

$$\Phi_0 = \begin{pmatrix}\Omega_0\\ -\Omega_0\end{pmatrix}.$$

Then it is easy to see that

$$[\Phi_0]^\perp = \left\{\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} \in \mathcal{H}\,\middle|\,\Psi_1^{(0)} = \Psi_2^{(0)}\right\},$$

where we write $\Psi_j = \{\Psi_j^{(n)}\}_{n=0}^\infty \in \mathcal{F}, \Psi_j^{(n)} \in \otimes_s^n L^2(\mathbb{R}^\nu)$. Let $\Psi \in [\Phi_0]^\perp$. Then, by the fact $H_b\Omega_0 = 0$ and $H_b \restriction \otimes_s^n L^2(\mathbb{R}^\nu) \ge nm$, we have

$$(\Psi, I \otimes H_b\Psi) \ge \sum_{j=1}^2\sum_{n=1}^\infty(\Psi_j^{(n)}, H_b\Psi_j^{(n)}) \ge m\sum_{j=1}^2\sum_{n=1}^\infty||\Psi_j^{(n)}||^2.$$

Noting the fact $\Psi_1^{(0)} = \Psi_2^{(0)}$, we have

$$\begin{aligned}
\frac{\mu}{2}(\Psi, \sigma_x \otimes I\Psi) &= \frac{\mu}{2}\{(\Psi_1, \Psi_2) + (\Psi_2, \Psi_1)\} \\
&= \frac{\mu}{2}\{|\Psi_1^{(0)}|^2 + |\Psi_2^{(0)}|^2\} + \frac{\mu}{2}\sum_{n=1}^{\infty}\{(\Psi_1^{(n)}, \Psi_2^{(n)}) + (\Psi_2^{(n)}, \Psi_1^{(n)})\} \\
&\geq \frac{\mu}{2}\{|\Psi_1^{(0)}|^2 + |\Psi_2^{(0)}|^2\} - \mu\sum_{n=1}^{\infty}\|\Psi_1^{(n)}\|\|\Psi_2^{(n)}\| \\
&\geq \frac{\mu}{2}\{|\Psi_1^{(0)}|^2 + |\Psi_2^{(0)}|^2\} - \frac{\mu}{2}\|\Psi\|^2.
\end{aligned}$$

These estimates and Lemma 5.4 give

$$\begin{aligned}
(\Psi, H\Psi) &\geq m(1-\varepsilon)\sum_{j=1}^{2}\sum_{n=1}^{\infty}\|\Psi_j^{(n)}\|^2 + \frac{\mu}{2}\{|\Psi_1^{(0)}|^2 + |\Psi_2^{(0)}|^2\} - \frac{\mu}{2}\|\Psi\|^2 - D_\varepsilon(\alpha, \mu)\|\Psi\|^2 \\
&\geq \left\{M_\varepsilon - \frac{\mu}{2} - D_\varepsilon(\alpha, \mu)\right\}\|\Psi\|^2,
\end{aligned}$$

where ε is an abitrary constant satisfying $0 < \varepsilon < 1$ and $M_\varepsilon = \min\left\{m(1-\varepsilon), \frac{\mu}{2}\right\}$. Since this inequality holds for all $\Psi \in [\Phi_0]^\perp$, we obtain $\mu_2(H) \geq C_0$ with

$$C_0 = \sup_{0<\varepsilon<1}\left\{M_\varepsilon - \frac{\mu}{2} - D_\varepsilon(\alpha, \mu)\right\}.$$

This estimate and the min-max principle imply that $E(H)$ is a simple eigenvalue of H if $E(H) < C_0$. By (2.10), if $C_0 > -\mu e^{-2\alpha^2\|\lambda/\omega\|^2}/2$ (this condition is equivalent to condition (1.11)), then $E(H) < C_0$ and hence H has a unique ground state. Thus the desired result follows.

5.3. *Proof of Theorem 1.3*

Let

$$\mu_3(H) = \sup_{\Phi_1, \Phi_2 \in \mathcal{H}} U_H(\Phi_1, \Phi_2)$$

with $U_H(\Phi_1, \Phi_2) = \inf_{\Psi \in D(H); \|\Psi\|=1, \Psi \in [\Phi_1, \Phi_2]^\perp}(\Psi, H\Psi)$, where $[\Phi_1, \Phi_2]^\perp$ denotes the orthogonal complement of $\{\alpha\Phi_1 + \beta\Phi_2 | \alpha, \beta \in \mathbb{C}\}$. Let

$$\Phi_1 = \begin{pmatrix} \Omega_0 \\ \Omega_0 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} \Omega_0 \\ -\Omega_0 \end{pmatrix}.$$

Then we have

$$[\Phi_1, \Phi_2]^\perp = \mathbb{C}^2 \otimes \mathcal{G} = \mathcal{G} \bigoplus \mathcal{G}$$

with $\mathcal{G} = \bigoplus_{n=1}^{\infty} \otimes_s^n L^2(\mathbb{R}^\nu)$. For all $\Psi = (\Psi_+, \Psi_-) \in [\Phi_1, \Phi_2]^\perp$ ($\Psi_\pm \in \mathcal{G}$), we have

$$(\Psi, H\Psi) \geq (\Psi_+, H_b\Psi_+) + (\Psi_-, H_b\Psi_-) - \frac{\mu}{2}||\Psi||^2.$$

It is easy to see that $(\Psi_\pm, H_b\Psi_\pm) \geq m||\Psi_\pm||^2$. Hence we obtain $(\Psi, H\Psi) \geq (m - \frac{\mu}{2})||\Psi||^2$, which implies that

$$\mu_3(H) \geq m - \frac{\mu}{2}. \tag{5.2}$$

Assume (1.12). Then, by (5.2) and (2.10), we have

$$\mu_3(H) > -\frac{\mu}{2}e^{-2||\lambda/\omega||_{L^2}^2} \geq E(H).$$

Hence, by the min-max principle, there are at most two eigenvalues (counting mutiplicity) of H in the interval $[E(H), -\frac{\mu}{2}e^{-||\lambda/\omega||_{L^2}^2}]$. In particular, H has at most two ground states. These facts and (2.5) imply Theorem 1.3. ∎

5.4. *Proof of Theorem 1.4*

We apply the following fact (which may be more or less known):

LEMMA 5.5. *Let $A_n, n = 1, 2, \cdots$, and A be self-adjoint operators on a Hilbert space $\mathcal{K}$ having a common core D such that, for all $\psi \in D$, $A_n\psi \to A\psi$ as $n \to \infty$. Let ψ_n be a normalized eigenvector of A_n with eigenvalue E_n: $A_n\psi_n = E_n\psi_n$ such that $E := \lim_{n\to\infty} E_n$ and $\mathrm{w}-\lim_{n\to\infty} \psi_n = \psi \neq 0$ exist, where $\mathrm{w}-\lim$ denotes weak limit. Then ψ is an eigenvector of A with eigenvalue E. In particular, if ψ_n is a ground state of A_n, then ψ is a ground state of A.*

PROOF: By the present assumption and a general theorem [10, Theorem VIII.25(a)], A_n converges to A in the strong resolvent sense as $n \to \infty$. Hence, for all $\phi \in \mathcal{K}$ and $z \in \mathbb{C}\setminus\mathbb{R}$, we have

$$\begin{aligned}
&|(\phi, (A_n - z)^{-1}\psi_n) - (\phi, (A - z)^{-1}\psi)| \\
&= |((A_n - z^*)^{-1}\phi - (A - z^*)^{-1}\phi, \psi_n)| + |((A - z^*)^{-1}\phi, \psi_n - \psi)| \\
&\leq ||(A_n - z^*)^{-1}\phi - (A - z^*)^{-1}\phi|| + |((A - z^*)^{-1}\phi, \psi_n - \psi)| \\
&\to 0 \quad (n \to \infty),
\end{aligned}$$

i.e., $\lim_{n\to\infty}(\phi, (A_n - z)^{-1}\psi_n) = (\phi, (A - z)^{-1}\psi)$. By the spectral theorem, we have $(\phi, (A_n - z)^{-1}\psi_n) = (E_n - z)^{-1}(\phi, \psi_n)$. Hence we obtain $(\phi, (A - z)^{-1}\psi) = (\phi, (E - z)^{-1}\psi)$ for all $\phi \in \mathcal{K}$, which implies that $(A - z)^{-1}\psi = (E - z)^{-1}\psi$. Thus $\psi \in D(A)$ and $A\psi = E\psi$. If ψ_n is a ground state of A_n, then $(\phi, A_n\phi) \geq E_n||\phi||^2$ for all $\phi \in D$. Taking the limit $n \to \infty$ in this inequality, we obtain $(\phi, A\phi) \geq E||\phi||^2$. Since D is a core for A, the last inequality extends to all $\phi \in D(A)$, which, combined with the preceding result, implies that $E = \inf \sigma(A)$. Thus ψ is a ground state of A. ∎

We now turn to the spin-boson Hamiltonian in the case $\inf_{k\in\mathbb{R}^\nu}\omega(k)=0$. To employ the results in the case of massive bosons, we define for $m>0$

$$\omega_m(k)=\omega(k)+m.$$

Then (1.2) with ω replaced by ω_m holds for all $m>0$. We introduce

$$H_{\mathrm{SB}}(m)=\frac{1}{2}\mu\sigma_z\otimes I+I\otimes H_b(m)+\alpha\sigma_x\otimes(a(\lambda)^*+a(\lambda))$$

with $H_b(m)=d\Gamma(\omega_m)$.

LEMMA 5.6. *Let $\mathcal{D}=\mathbb{C}^2\hat{\otimes}[\mathcal{F}_0\cap D(H_b)]$, where $\hat{\otimes}$ denotes algebraic tensor product. Then $\mathcal{D}$ is a common core for all $H_{\mathrm{SB}}(m)$ and H_{SB}. Moreover, for all $\Psi\in\mathcal{D}$, $H_{\mathrm{SB}}(m)\Psi\to H_{\mathrm{SB}}\Psi$ as $m\to 0$.*

PROOF: The first half of the lemma is well known (note that $\mathbb{C}^2\hat{\otimes}[\mathcal{F}_0\cap D(H_b)]=\mathbb{C}^2\hat{\otimes}[\mathcal{F}_0\cap D(H_b(m))]$). The second half follows from a direct computation. ∎

We are now ready to prove Theorem 1.4. By Theorem 1.1, there exists a ground state $\Omega(m)$ of $H_{\mathrm{SB}}(m)$: $H_{\mathrm{SB}}(m)\Omega(m)=E(H_{\mathrm{SB}}(m))\Omega(m)$. Without loss of generality, we can assume that $||\Omega(m)||=1$. By (1.8), we have

$$-\frac{\mu}{2}-\alpha^2\left\|\frac{\lambda}{\sqrt{\omega_m}}\right\|_{L^2}^2\le E(H_{\mathrm{SB}}(m))\le -\frac{\mu}{2}e^{-2\alpha^2||\lambda/\omega_m||_{L^2}^2}-\alpha^2\left\|\frac{\lambda}{\sqrt{\omega_m}}\right\|_{L^2}^2.$$

By using the Lebesgue dominated convergence theorem, one casn easily show that

$$\lim_{m\to 0}\left\|\frac{\lambda}{\sqrt{\omega_m}}\right\|_{L^2}^2=\left\|\frac{\lambda}{\sqrt{\omega}}\right\|_{L^2}^2,\quad \lim_{m\to 0}\left\|\frac{\lambda}{\omega_m}\right\|_{L^2}^2=\left\|\frac{\lambda}{\omega}\right\|_{L^2}^2. \tag{5.3}$$

Hence $\{E(H_{\mathrm{SB}}(m))\}_m$ is uniformly bounded in m. Thus there exists a sequence $\{m_j\}_{j=1}^\infty$ with $m_1>m_2>\cdots>m_j\to 0$ $(j\to\infty)$ such that

$$E:=\lim_{j\to\infty}E(H_{\mathrm{SB}}(m_j))$$

and

$$\Omega:=\mathrm{w}-\lim_{j\to\infty}\Omega(m_j)$$

exist. We need only to show that $\Omega\neq 0$ (then, by Lemmas 5.6 and 5.5, Ω is a ground state of H_{SB}).

Let P_0 be the orthogonal projection from $\mathcal{F}$ onto the Fock vacuum state $\{c\Omega_0|c\in\mathbb{C}\}$. It is easy to see that

$$I\otimes P_0\ge I-I\otimes N.$$

If $\omega\lambda$ and λ are in $L^2(\mathbb{R}^\nu)$, then $\omega_m\lambda \in L^2(\mathbb{R}^\nu)$. By these facts and Lemma 2.3, we have

$$(\Omega(m), I \otimes P_0\Omega(m)) \geq 1 - (\Omega(m), I \otimes N\Omega(m)) \geq 1 - \alpha^2 \left\| \frac{\lambda}{\omega_m} \right\|_{L^2}^2 . \tag{5.4}$$

Since the range of $I \otimes P_0$ is finite dimensional (in fact, two dimensional), we have

$$\lim_{j\to\infty} (\Omega(m_j), I \otimes P_0\Omega(m_j)) = (\Omega, I \otimes P_0\Omega).$$

From this fact, (5.4) and the second formula in (5.3), we obtain

$$(\Omega, I \otimes P_0\Omega) \geq 1 - \alpha^2 \left\| \frac{\lambda}{\omega} \right\|_{L^2}^2 .$$

Under condition (1.13), the RHS is strictly positive. Hence $\Omega \neq 0$. ∎

6. A generalization of the model

In this section we propose a generalization of the spin-boson model discussed in the preceding sections. We expect that the generalization clarify the general properties of the spin-boson model. We also have in mind applications to quantum spin systems on an infinite lattice in which spins interact with bosons too.

Let $\mathcal{H}$ be a Hilbert space and A (resp. B) be a self-adjoint (resp. symmetric) operator on $\mathcal{H}$. The Hamiltonian of the genelaized spin-boson model we propose is given by

$$H = A \otimes I + I \otimes d\Gamma(\omega) + B \otimes (a(\lambda)^* + a(\lambda))$$

acting in the Hilbert space $\mathcal{H} \otimes \mathcal{F}$.

Supposse that A, B are bounded and $\lambda, \lambda/\sqrt{\omega}, \lambda/\omega$ are in $L^2(\mathbf{R}^d)$. Then

$$L_{A,B} := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i||\lambda/\omega||_{L^2}Bt} A e^{i||\lambda/\omega||_{L^2}Bt} e^{-t^2/2} dt - \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_{L^2}^2 B^2$$

is a bounded self-adjoint operator. We can show [4] that

$$-||A|| - ||B||^2 \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_{L^2}^2 \leq E(H) \leq E(L_{A,B}). \tag{6.1}$$

In the case of the original spin-boson model (i.e., the case $H = H_{\text{SB}}$), (6.1) is just (1.8). Thus estimate (6.1) clarifies a general structure of (1.8). The results on ground states of H_{SB} also can be generalized to the case of H. We can also develop scattering theory concerning the pair $< A \otimes I + I \otimes d\Gamma(\omega), H >$. For the details, see [4].

References

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